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Newton's *Principia* and Inverse-Square Orbits in a Resisting Medium: A Spiral of Twisted Logic

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Proposition XV/Theorem XII in Book Two of Newton's *Principia* deals with the spiral path of a body attracted by an inverse-square force toward a fixed center and retarded by the medium in which it travels. This article examines the argument offered by Newton as proof of the proposition/theorem and finds it fallacious. Also presented here are accounts of how Newton's purported proof is dealt with in each of three late-20th-century publications—none of which reports detection of the fallacy. © 1998 Academic Press

La Proposition XV (théorème XI) du livre II des *Principia* de Newton traite de la trajectoire en forme de spirale d'un corps attiré vers un centre immobile par une force d'intensité proportionnelle à l'inverse du carré de la distance, et ralenti par le milieu qu'il traverse. Cet article examine le raisonnement offert par Newton pour preuve de la proposition (du théorème) et montre qu'il est incorrect. L'article présente aussi la façon dont la prétendue preuve de Newton a été traitée dans trois publications de la fin du vingtième siècle—aucune d'entres elles ne découvre l'erreur du raisonnement. © 1998 Academic Press

Proposition XV/Theorem XII der *Principia* Buch Zwei von Newton befaßt sich mit dem spiralförmigen Weg eines Körpers der durch einer Kraft gemäß dem inversen Quadratgesetz zu einen bestimmten Mittlepunkt, und durch das Medium verzögert wird, worin er sich bewegt. Dieser Aufsatz untersucht das von Newton vorgebrachte Argument für den Beweis den Satzes/Theorems, und findet es durch einen Trugschluß beeinträchtigt. Auch wird hier dargelegt, wie der von Newton gegebene Beweiß in jeden der drie Veröffentlichungen des späten zwanzigsten Jahrhunderts behandelt wird—keiner von ihnen berichtet von der Entdeckung des Trugschlußes. © 1998 Academic Press

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INTRODUCTION

At least three separate beams of attention have been directed, in recent years, upon Proposition XV/Theorem XII in Book Two of Newton's *Principia*—hereinafter denoted 2XV—and upon the argument offered by Newton as proof of 2XV [13, 282–284]. This essay focuses upon the most remarkable feature of Newton's argument and aims to illuminate the particular way in which each of the three beam directors deals with it. Indeed, it is established in what follows that Newton's argument is palpably fallacious—a fact not even hinted at by any of the three, all of whom treat the argument as if it were valid.

KING-HELE AND NEWTON

The earliest and most eye catching of the three recently published foci upon 2XV reached this author's attention in a review [6] of Desmond King-Hele's autobiography, *A Tapestry of Orbits* [7]. The review reports, among other things, that King-Hele "was particularly pleased to discover that a result he had established in 1956 for the descent of a satellite through the earth's atmosphere was not new, but had been derived by Newton in the *Principia* ..." [6, 377]. From the autobiography itself, one learns that the matter is given detailed treatment in [9], in which 2XV is identified as providing the derivation that King-Hele supposed had first been achieved by himself and a collaborator [8].

Using the standard English version [13, 282] of the third edition (1726) of the *Principia*, [9, 270] presents 2XV verbatim:

> PROPOSITION XV. THEOREM XII. If the density of a medium in each place thereof be inversely as the distance of the places from an immovable centre, and the centripetal force be as the square of the density: I say, that a body may revolve in a spiral which cuts all the radii drawn from that centre in a given angle.

A "spiral which cuts all the radii drawn from [a] centre in a given angle" is an *equiangular spiral*.

Next, we read "[w]e shall not go through Newton's proof in detail, firstly because you can read it in the *Principia* and secondly because it is difficult to follow. Instead, here is a very simple, though not rigorous, modern derivation of his result" [9, 270]. It is not clear as to whether the joke in the first of the two sentences quoted is intentional. But there is no joke involved in stating here the fact that what Newton offers as a proof of 2XV is an undisguised example of a fallacy the precise form of which emerges below. In order to maximize the ease of its perception by the reader and also to minimize the typesetting requirements for its illumination, here are alphabetic abbreviations of statements respectively equivalent to the three found in 2XV and to one omitted therefrom but implicitly involved therein:

A = "density of medium is proportional to $(1/r)$,"
B = "centripetal force is proportional to $(1/r^2)$,"
C = "tangential resisting force is proportional to ρv^2," and
E = "orbit is an equiangular spiral centered at $r = 0$,"

where r is the radial coordinate measured from a fixed origin ("the distance of the places from an immovable centre"), ρ is the density of the ambient medium at the body's position, and v is the body's instantaneous speed. Although not explicitly mentioned in 2XV, assumption C is implicit therein, since Newton uses it in his argument that purports to furnish a proof of 2XV. (One notices that B expresses the variability of the centripetal force—"as the square of the density"—in terms of r by means of A.) We can therefore set down the equivalence

$$2\text{XV} = [(\text{A, B, and C}) \Rightarrow \text{E}],$$

in which the expression between brackets reads "if all of A, B, and C obtain, then E must (or will) result."

(The actual wording of 2XV in [13, 282] uses the verb "may" in place of the "must" or "will" one normally expects in the statement of a theorem. The indefinite "may" makes no sense in modern usage; we must conclude that Newton intended what we today would express as "must" or "will." It is the Latin "potest" [10, 409] that is translated "may" in [12, 223] by Andrew Motte and maintained by Florian Cajori in [13, 282] for the respective English statements of 2XV; but in the final sentence of Newton's putative proof of 2XV the same "potest" [10, 411] is rendered as "will" by Motte and is likewise kept by Cajori.)

That "Newton's (purported) proof . . . is difficult to follow" is an understatement. Yet, as we see directly, the portion of it that constitutes fallacy protrudes from the turbid bulk with unmistakable clarity. How this is so emerges from the following sequence:

(i) Directly preceding 2XV is Lemma III, whose statement begins "Let PQR be a spiral cutting the radii SP, SQ, SR, &c, in equal angles" [13, 282]. It thus deals with an equiangular spiral and continues to deal with this through to its conclusion, which expresses a geometric property of the spiral that is of no concern to us here.

(ii) The argument offered as proof of 2XV [13, 283–284] commences with "Suppose everything to be as in the foregoing Lemma. . . . In any time let a body, in a resisting medium, describe [an arc of the spiral]. . . ." In brief, the argument begins with supposition of E—the very conclusion of the theorem 2XV it claims to be proving.

(iii) We further observe that the body, assumed to be moving along the equiangular spiral in a resisting medium that exerts the opposing tangential force proportional to ρv^2, is also subject to an inverse-square attraction toward the point about which the spiral is described. That is, B and C are also assumed.

(iv) Step after step—geometric, kinematic, dynamic—the intricate process plods its wearying way, as the reader can readily verify. There is no need in the present context to check the validity of each of the many individual steps that constitute the unfolding argument. We must, however, take meticulous care to examine every one of them in order to make certain that the argument does not stray from dependence on its primary suppositions B, C, and E. We make this check upon statement after statement, without missing a step, line after line, the entire length of p. 283 onto p. 284 of [13] through the sentence that ends, nearly halfway down p. 284, "and there will remain the density of the medium in P, as $OS/OP \cdot SP$." Here, the distances OS, OP, and SP are measured in relation to the posited equiangular spiral with which the argument under our scrutiny begins. Indeed $SP = r$, the distance from the force center S to the arbitrary point P of the spiral.

(v) The next sentence begins "Let the spiral be given . . . ," which we have taken care to remember all along; so let the sentence continue ". . . and, because of the given ratio of OS to OP, the density of the medium in P will be as 1/SP." That is to say, it has been deduced that the density of the retarding medium at distance $r = SP$ from the force center must be proportional to $(1/r)$.

(vi) Recognizing the final sentence of (v) as a statement of A, and recalling (ii) and (iii) along with the meticulous check described in (iv), we can summarize the *Principia* argument thus far as having reached

$$(\text{B, C, and E}) \Rightarrow \text{A}.$$

(vii) Then, in the very next sentence after the one brokenly quoted in (v) above, Newton concludes his proffered proof of 2XV with "Therefore, in a medium whose density is inversely as SP the distance from the centre, a body will revolve in this spiral. Q.E.D." That is, Newton claims

$$(\text{A, B, and C}) \Rightarrow \text{E}$$

—namely, 2XV—as a direct consequence of (B, C, and E) ⇒ A, the result achieved in the course of (ii) through (v). An elementary example shows that the final step of the foregoing argument is fallacious, for if it were valid an absurdity must result:

Let n stand for a real number and set the following definitions:

C = "n is an integer,"
A = "n is odd,"
B = "n is greater than 2," and
E = "n is a prime."

(We recall that a prime is a positive integer that has exactly two distinct divisors: 1 and itself.) Thus

$$(\text{B, C, and E}) \Rightarrow \text{A}$$

reads "if integer n is a prime greater than 2, then n is odd"—a well known fact, since 2 is the only even prime. If—as the final step of the *Principia*'s proffered argument in support of 2XV declares—

$$(\text{A, B, and C}) \Rightarrow \text{E}$$

were actually a logical consequence of (B, C, and E) ⇒ A, we should then conclude that "if integer n is odd and greater than 2, then n is prime"—an absurdity since 15, for example, is odd yet not a prime.

Another way of perceiving the fallacy embodied in the *Principia* argument outlined in (ii)–(vii) above is to symbolize it as

$$[(\text{B, C, and E}) \Rightarrow \text{A}] \Rightarrow [(\text{A, B, C}) \Rightarrow \text{E}]$$

and—instead of using a particular example to establish the logical absurdity of the central implication arrow—we observe an open violation of an important principle of logic: *One must never, as part of a proof, assume and make use of a statement that one intends to arrive at as conclusion; introduction and use of such a statement—tantamount to assuming what one seeks to prove—renders a purported proof fallacious* [3, 5: 64]. The introduction and use of E in order to arrive at E as conclusion necessarily invalidates the argument expressed in the one line of symbols directly above. That this is so was surely as self-evident to practitioners of mathematics from the age of classical Greek geometry onward as it is today. If explicit evidence were required to establish the principle as being securely in place by the time of Isaac Newton, one could point to Johann Bernoulli's justified objection in 1710 to what the first (1687) edition of the *Principia* had erroneously offered as proof, in Propositions XI–XIII *cum* Corollary 1 of Book One, that inverse-square force implies conic-section orbit: The fallacy exposed by Bernoulli is at base a violation of the principle here cited [16]. (See also [11, 36].) As concluded in [20, 196], "in the 17th century ... an adequate basis for mathematics, accepted as a matter of practice, did exist which was little different, if at all, from that ... in Greek and medieval times."

Some readers—perhaps all—may have early on perceived the malfeasance in the *Principia* argument while reading the outline of it in (i)–(vii) above: the assumption and use of E = "orbit is an equiangular spiral centered at $r = 0$" as part of what Newton offers as a proof that reaches E as conclusion. In any event, the pleasure reportedly found by King-Hele in being anticipated by Newton might well be replaced by gratification in his knowing that he and Doreen Gilmore were, after all, the first to achieve a proof of Proposition XV/Theorem XII in Book Two of Newton's *Principia*.

The fallacious argument offered by Newton in 2XV is, incidentally, the same in all three *Principia* editions (1687, 1713, 1726) published during Newton's lifetime [10,1: 409–411]. (A slight modification of two lines from the first edition to the second entails no change in the argument's structure.) I am convinced that such an obvious elementary error went unreported until late in the 20th century because of the painful unreadability of very much of the *Principia*. In particular, details of the reasoning referred to above that yields the conclusion

$$(\text{B, C, and E}) \Rightarrow \text{A}$$

are so noisomely intricate as to distract the reader from detecting the fault in the overall argument. Such distraction must have beset King-Hele, Walker, and Chandrasekhar (cited directly below), for example. It is noteworthy in this connection that even the quite obvious fallacy perpetrated in Propositions XI–XIII *cum* Corollary 1 of Book One was not reported by Johann Bernoulli until 1710, some 23 years after its publication in the first edition [16].

Newton's thought on the matter at hand is not accessible to us, but it is difficult to suppose that he was unaware of the logical principle violated in the argument he presented as proof of 2XV. Can it then be possible that the Cambridge professor was aware of his inability to prove 2XV and therefore presented an intricacy-infested counterfeit proof while entertaining the hopeful expectation that no one would detect the fallacy? Scholars in our time have pointed to items in the *Principia* that betray the aroma of swindle knowingly committed by the tome's author. According to [5, 12], "[Johann] Bernoulli ... found other places in the *Principia* where Newton, in difficulty, talked what Bernoulli called 'gibberish' in order to try to dodge problems ..." [1, fols. 3v–4r]. (See also [11, 35–36; 19; 15, 200–201].)

CHANDRASEKHAR

In *Newton's Principia for the Common Reader*, published shortly before his death in 1995, Subrahmanyan Chandrasekhar begins his treatment of 2XV (and of a generalization—inverse nth-power central-force attraction—found in Proposition XVI): "In Propositions XV and XVI, Newton formulates a problem on the effect of air-drag on the descent of bodies under centripetal attraction. As a prelude to Newton's method of solution, we shall state the problem simply and provide the solution as one might today" [2, 539]. We find, however, no statement, simple or otherwise, of the problem being solved. There is merely a step-by-step derivation that uses as given the central-attraction force law (γ/r^n) and the tangential retardation acceleration (Dv^2/r), where n, γ, and D are constants, and v is the speed of the orbiting particle. Then, well along in the derivation—still without a statement of the problem being solved—Chandrasekhar introduces the additional assumption that the orbit is an equiangular spiral and proceeds to achieve a formula for D in terms of the constant angle specifying the assumed orbit—as his *end result*. We recall that even for $n = 2$, the inverse-square case, this is not at all the problem Newton claims to solve in 2XV (or even that which one might give him credit for solving if one extracted from 2XV the rational portion thereof). Yet Chandrasekhar writes "Newton obtains the foregoing solutions in Propositions XV and XVI ..." [2, 541].

He next [2, 541–545] offers verbatim the statements of Lemma III referred to above and Proposition XV/Theorem XII—namely, 2XV—separated only by his paraphrase, in full detail, of the *Principia* proof of Lemma III. Then, following the statement of 2XV, Chandrasekhar produces a step-by-step trace, again in paraphrase, of Newton's argument in support of 2XV all the way through the penultimate statement—expressed in *Principia* language under (v) above—in his own words: "Therefore the density must vary inversely as r if the orbit is to be an equi-angular spiral" [2, 545]. And, without finding a gap, we next read with amazement "As Newton concludes: Therefore in a medium whose density is inversely as SP [= r] the distance from the centre, a body will revolve in this spiral. Q.E.D." [2, 545]. Instead of calling attention to the blemish wrought by Newton's *non sequitur*, Chandrasekhar repeats it as if it were valid!

ERLICHSON

In an article [4] published before the appearance of Chandrasekhar's tome, Herman Erlichson also presents a verbatim statement of 2XV, to which he directly appends this remark: "Note carefully that Newton is not claiming that the equiangular spiral is the only orbit, he is saying that it is at least one of the possible orbits for the given force condition" [4, 282]. This remark evidently connotes an interpretation of 2XV somewhat as follows: Under the hypotheses A, B, and C of 2XV, there must be certain examples of ancillary circumstances (initial conditions, perhaps) whose particular incidence will require that orbit to be an equiangular spiral, although not necessarily all examples of the circumstances will yield such an orbit. (Erlichson's use of "one of the possible orbits" surely reflects the appearance of the auxiliary verb "may"—instead of "must" or "will" as noted in the parenthesized paragraph above—in his rendition of 2XV [13, 282], as quoted above from [9, 270].)

After an explicit statement (merely implicit under 2XV in the *Principia*) that the retarding force "goes as the product of the density of the medium and the square of the velocity," Erlichson presents an expanded, carefully didactic, version of Newton's argument that appears directly below the statement of 2XV in the *Principia*. He proceeds, however, only through the deduced statement, found under (v) above, that the density of the retarding medium is proportional to (1/SP) = $(1/r)$. But next, instead of repeating Newton's *non sequitur* (as does Chandrasekhar), he introduces one of his own that is an echo of his remark quoted above: "Thus, if we have a medium where the density goes as $1/r$, and a centripetal force which goes as the square of the density (goes as $1/r^2$), and a resistive force which goes as the product of the density and the square of the velocity, then an equiangular spiral is a *possible* orbit" (emphasis in original) [4, 290].

Erlichson's fallacy differs from Newton's, we observe, merely in its replacement of "a body will revolve in this spiral" by "then an equiangular spiral is a *possible* orbit"—both quoted conclusions based on the single argument that uses, as an *assumption*, motion along an equiangular spiral. One therefore cannot deny the fallacious character of the conclusion whose statement ends "then an equiangular spiral is a *possible* orbit."

To set into symbolic arrangement the formal establishment of Erlichson's argument as fallacious, we identify it as use of the heterodox implication

$$[(\text{B, C, and E}) \Rightarrow \text{A}] \rightarrow [(\text{A, B, and C}) \Rightarrow \text{E}],$$

in which A, B, C, E have the respective meanings assigned to them above in reference to 2XV, and "→" can be read "implies, in at least one circumstance." To show that the heterodox implication is *false*, let it be applied to the case in which

$$\text{B} = \text{“}x \text{ is a real number,”}$$

$$\text{A} = \text{“}x = 2 \text{ or } x = -2\text{,”}$$

$$C = \text{“}3 \le x^2 \le 5\text{,” and}$$

$$\text{E} = \text{“}\sqrt{x^2-2} - \sqrt{6-x^2} - \sqrt{2}(\sqrt{x^2-3} + \sqrt{5-x^2}) = 0\text{,”}$$

where $\sqrt{p}$ represents the *nonnegative* square root of p for all $p \geq 0$. Following the rules of elementary algebra, we observe that

$$
\begin{aligned}
(\text{B, C, and E}) &\Rightarrow (\sqrt{x^2-2} - \sqrt{6-x^2})^2 = 2(\sqrt{x^2-3} + \sqrt{5-x^2})^2 \\
&\Rightarrow 4 - 2\sqrt{(x^2-2)(6-x^2)} = 2[2 + 2\sqrt{(x^2-3)(5-x^2)}] \\
&\Rightarrow -\sqrt{(x^2-2)(6-x^2)} = 2\sqrt{(x^2-3)(5-x^2)} \\
&\Rightarrow -x^4 + 8x^2 - 12 = 4(-x^4 + 8x^2 - 15) \\
&\Rightarrow 3(x^4 - 8x^2 + 16) = 0 \Rightarrow (x^2-4)^2 = 0 \\
&\Rightarrow x = 2 \text{ or } x = -2.
\end{aligned}
$$

That is, (B, C, and E) ⇒ A. Now, can we infer from this that there is some circumstance in which (A, B, and C) leads to E as consequence? Erlichson's argument says there is at least one; yet we calculate to ascertain that

$$\begin{aligned}(\text{A, B, and C}) &\Rightarrow \sqrt{x^2-2}-\sqrt{6-x^2}-\sqrt{2}(\sqrt{x^2-3}+\sqrt{5-x^2})\\ &= \sqrt{4-2}-\sqrt{6-4}-\sqrt{2}(\sqrt{4-3}+\sqrt{5-4})\\ &= -2\sqrt{2}\neq 0.\end{aligned}$$

Thus there is no circumstance in which E is a consequence of (A, B, and C).

RELATED ITEMS

There is just one other publication known to the present writer in which 2XV is considered. In [21, 6: 357–358], the editor devotes a brief portion of his note 217 to mathematical matters related to 2XV; but he neither states the proposition/theorem nor describes the argument that Newton offers as proof of it. In particular, he mentions nothing of the fallacy that vitiates the argument.

The present author is aware of two other examples of *Principia*-borne fallacies having the same character as the one exposed above in this article:

(1) Corollary 5 to Proposition VI in Book One [13, 48–49] is presented *as a consequence* of Proposition VI itself or of any one of its first four corollaries, all equivalent to it. Corollary 5 is, in fact, a *converse* of the proposition from which it is said to follow. One readily perceives that such an inference is a clear-cut violation of the simple basic principle of logic presented in italics above and in [3, 5: 64]. A detailed analysis of the fallacy in Corollary 5 can be found in [17, 66].

(2) The argument embodied in Propositions XI–XIII *cum* Corollary 1 in Book One [13, 56–61] purports to prove (at least in outline) that inverse-square central force implies conic-section orbit; yet it uses *as hypothesis* a particle moving in a conic-section orbit. This violation of the same italicized basic principle is for some scholars expunged by a pair of sentences appearing only in the third (1726) edition of the *Principia*; but the sentences do not in fact remove the fallacy according to others, this author included. A full analysis is provided in [18, 185–187].

AN OPINION

The *Principia's* status as the most widely and lavishly praised of all books on physical science makes imperative, in my opinion, exposure to the scholarly world of each nontrivial fault found between its covers—especially when there are recent publications that treat the fault as if it were free of disabling error.

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This essay would not have been written but for Derek Gjertsen, who directed my attention, in September 1995, to the review [6] of [7]. Although I had previously studied several other sections of Newton's *Principia*, I had never examined 2XV before being led to it by King-Hele's erroneous account [9, 270] of its having anticipated his and Gilmore's 1956 derivation. I wish also to thank John Greenberg for alerting me to the existence of [4] after learning of my recently stimulated interest in 2XV. My grateful thanks go also to my friends Sigrid Boe and Booker Peek, who kindly produced the German and French translation of the abstract.

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